Exact Solutions for the L^1TV Functional

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Outline

- L^2 TV Functional $\rightarrow L^1$ TV Functional

- Properties of L^1 TV : $f = \chi_\Omega \Rightarrow u = \chi_\Sigma$ for some Σ.

- Properties of L^1 TV : For $f = \chi_\Omega$, $\int |\nabla u| + \lambda \int |u - f| = \mathrm{Per}(\Sigma) + \lambda |\Sigma \triangle \Omega|$

- Properties of L^1 TV: $\Omega = B_{\frac{2}{\lambda}}$ and non-uniqueness.

- Properties of L^1 TV: Preservation of contrast: for any C^2 set Ω there is a λ^* such that $\lambda \geq \lambda^*$ implies that $\Sigma = \Omega$.

- New Result 1: $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma$

- New Results 2: $B_{\frac{2}{\lambda}} \underset{\sim}{\subset} \Omega \rightarrow B_{\frac{2}{\lambda} - \epsilon} \subset \Sigma$

- New Result 3: A complete characterization of Σ for convex Ω.

Motivation

We are pursuing this research with the following objectives in mind:

- To understand precise nature of denoising/reconstruction we want to understand, in great detail, how methods deform/preserve simple images.

- Allows us to compare denoising methods and understand implicit biases.

- Moves us towards a precise and systematic approach to prior informed regularization for complex experimental images.

The classic Rudin-Osher-Fatemi (ROF) total variation regularized functional:

$$F(u) \equiv \int |\nabla u| dx + \lambda \int |u - f|^2 dx \tag{1}$$

is characterized by *simultaneous edge recovery/preservation and noise reduction* but also *loss of contrast*.

(I) *total variation of* $u = \int |\nabla u|$
(II) $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and $\int |\nabla u| < \infty$.

$$(\text{III}) \int |\nabla u| = \sup\{\textstyle\int \nabla u \cdot \vec{g} \; ; \; |\vec{g}| < 1, \vec{g} \in C_0^1(\Omega; \mathbb{R}^n)\} \tag{2}$$

$$= \sup\{\textstyle\int u \text{div} \vec{g} \; ; \; |\vec{g}| < 1, \vec{g} \in C_0^1(\Omega; \mathbb{R}^n)\} \tag{3}$$

for $u \in W^{1,1}$.

(IV) For $u \in L^1(\Omega)$, we use the last equation to *define* $\int |\nabla u|$

(V) The theory for $BV(\Omega)$ is extensive and quite beautiful [see for example Giusti's "Minimal Surfaces and Functions of Bounded Variation" and Evan's and Gariepy's "Measure Theory and Fine Properties of Functions"]

Other facts:

1. $TV(u)$ is lower semi-continuous in L^1

2. Approximation, compactness, and trace results are similar to Sobolev spaces.

3. $Du = \nabla u$ is a vector valued Radon measure

4. $\int |\nabla u| = \int_{u_{\min}}^{u_{\max}} \text{Per}(\{x \; ; \; u(x) > t\})dt$ (coarea formula)

$L^1\ TV$ Functional: $TV + L^1$ Data Fidelity

The $L^1\ TV$ functional, previously studied by [Alliney], [Nikolova] and [Chan and Esedoglu] is given by:

$$F(u) \equiv \int |\nabla u| dx + \lambda \int |u - f| dx \qquad (4)$$

and is characterized by *simultaneous edge recovery/preservation and noise reduction without the loss of contrast*. But this is not all:

1. $F(u)$ is not strictly convex $\Rightarrow$ we do not have uniqueness!

2. u is a minimizer for $f \rightarrow Cu$ is a minimizer for Cf

3. $f = \chi_\Omega \rightarrow u = \chi_\Sigma$.

We now look at the properties of $L^1\ TV$ a bit more closely.

Result:[Chan and Esedoglu] If u is any minimizer of $F_\lambda(u)$ then for almost all $\mu \in [0, 1]$,

$$\chi_{\{x:u>\mu\}}$$

is also a minimizer of $F_\lambda(u)$.

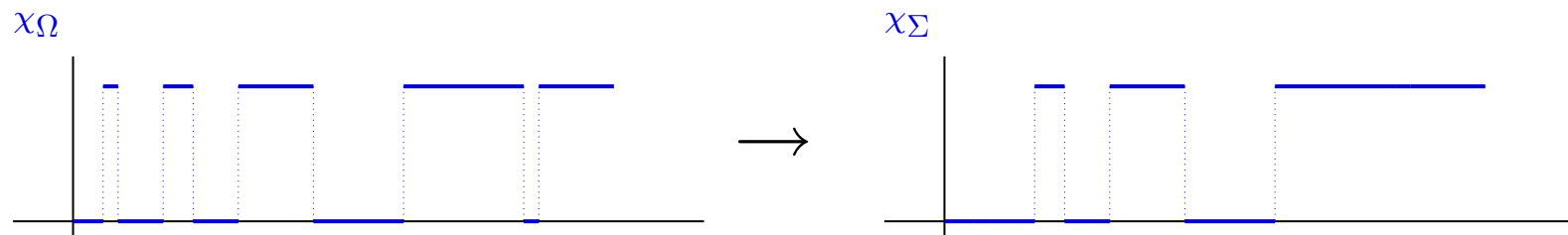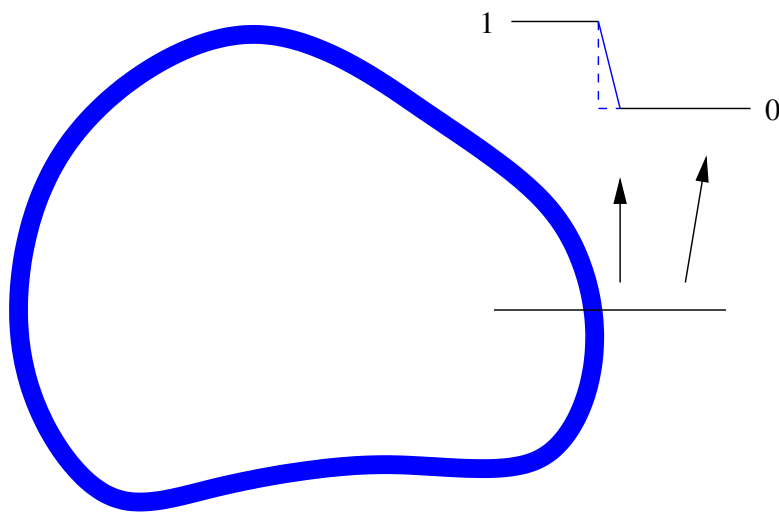One Dimensional Example: λ determines which interval of Ω appears in Σ.

Figure 1: Small segments disappear: λ determines "small"

Segment preserved if $\{\mathrm{Per}(I) = 2 < \lambda L_I \Leftrightarrow L_I > \frac{2}{\lambda}\}$.

For characteristic functions $u = \chi_\Sigma$ (binary images) $\int |\nabla u|$ is exactly the perimeter and $\lambda \int |u - f| = \lambda \int |\chi_\Sigma - \chi_\Omega| = \lambda \int |\Sigma \triangle \Omega|$.

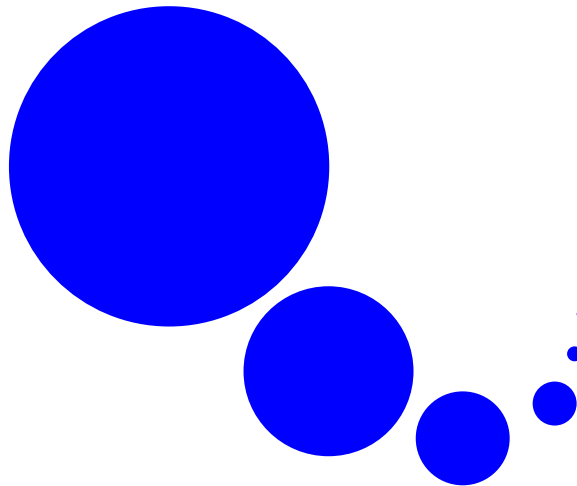$$\int |\nabla \chi_\Sigma| = \mathrm{Per}(\Sigma) \qquad \lambda \int |u - f| = \lambda \int |\chi_\Sigma - \chi_\Omega| = \lambda \int |\Sigma \triangle \Omega|$$
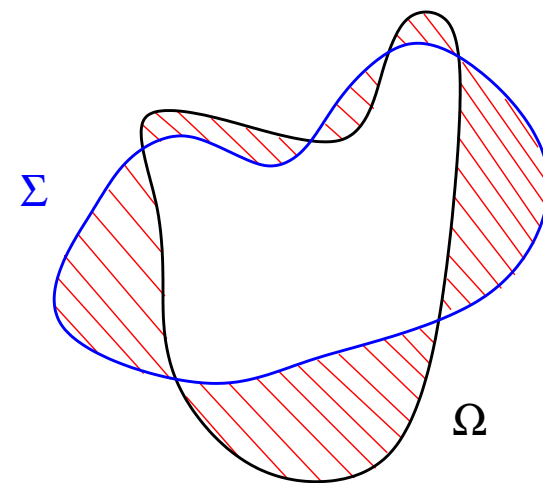
Result:[Chan and Esedoglu] If $\Omega = B_{\frac{2}{\lambda}}$ then $u = \alpha \chi_{B_{\frac{2}{\lambda}}}$ is a minimizer for any $\alpha \in [0, 1]$.

One can therefore concoct Ω's whose solutions $\Sigma(\lambda)$ have, as $\lambda \to \infty$, an infinite number of non-uniqueness points ...

The previous example demonstrated an Ω that is never reproduced by Σ as $\lambda \to \infty$. When Ω is a bounded, C^2 set this can't happen:

Result:[Chan and Esedoglu] For Ω bounded and C^2 there is a $\lambda^* < \infty$ such that for all $\lambda \geq \lambda^*$, $\Sigma = \Omega$.

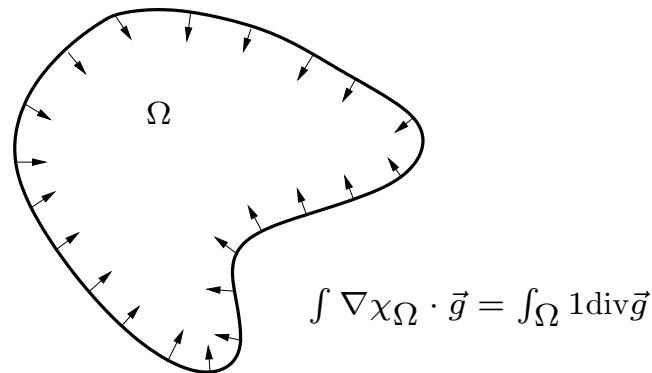Choose $\vec{g}$ such that $\int_\Omega 1 \mathrm{div}\vec{g} = \mathrm{Per}(\Omega)$:

$$\Omega$$

$$\int \nabla \chi_\Omega \cdot \vec{g} = \int_\Omega 1\mathrm{div}\vec{g}$$

Figure 2: choosing a vector field

L^1 TV: smooth Ω + big λ imply $\Sigma = \Omega$.

$$
\begin{aligned}
\mathsf{Per}(\Sigma) + \lambda|\Sigma \triangle \Omega| \;\geq\; & \int_\Sigma 1 \mathsf{div}\vec{g} + \lambda \int_{\Omega \cap \Sigma^c} 1 + \lambda \int_{\Omega^c \cap \Sigma} 1 && (5) \\
=\; & \int_{\Omega \cap \Sigma} 1 \mathsf{div}\vec{g} \;+\; \int_{\Omega^c \cap \Sigma} 1 \mathsf{div}\vec{g} + \lambda \int_{\Omega \cap \Sigma^c} 1 + \lambda \int_{\Omega^c \cap \Sigma} 1 && (6) \\
\geq\; & \int_{\Omega \cap \Sigma} 1 \mathsf{div}\vec{g} \;+\; \int_{\Omega^c \cap \Sigma} 1 \mathsf{div}\vec{g} + \int_{\Omega \cap \Sigma^c} 1 \mathsf{div}\vec{g} + \lambda \int_{\Omega^c \cap \Sigma} 1 && (7) \\
=\; & \int_\Omega 1 \mathsf{div}\vec{g} + \int_{\Omega^c \cap \Sigma} 1(\lambda + \mathsf{div}\vec{g}) && (8) \\
\geq\; & \mathsf{Per}(\Omega) && (9)
\end{aligned}
$$

as long as $\lambda > ||\mathsf{div}\vec{g}||_\infty$.

Theorem 1. *If $B_r \subset \Omega$ where $r \geq \frac{2}{\lambda}$, then $B_r \subset \Sigma$.*

In particular, we can conclude that the boundary of Σ is in the envelope of inside and outside $\frac{2}{\lambda}$ balls.
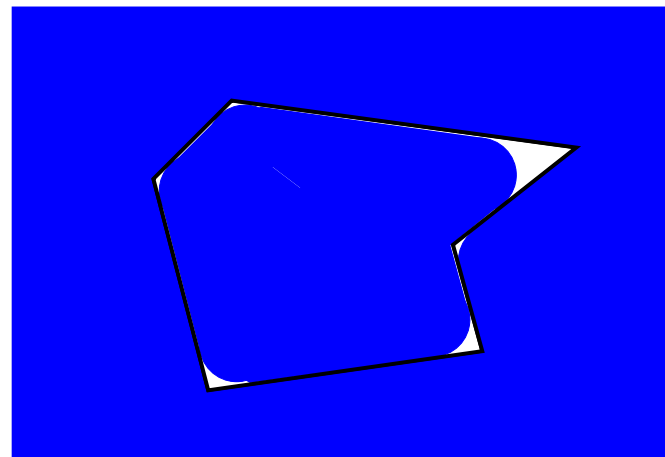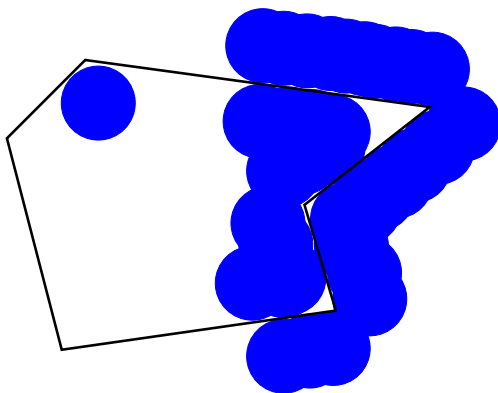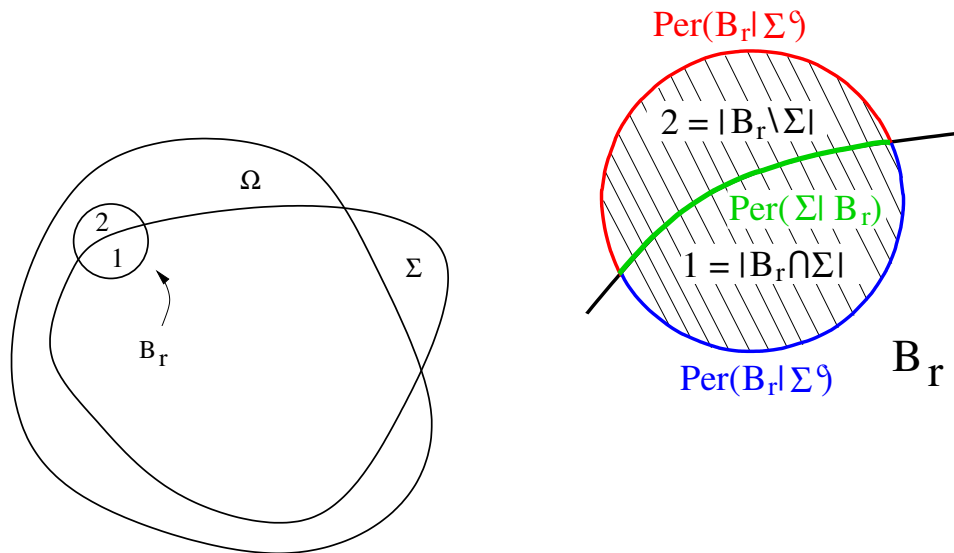
$$E(\Sigma \cup B_r) - E(\Sigma) \quad = \quad (\mathsf{Per}(B_r) - \lambda|B_r|) + (\lambda|B_r \cap \Sigma| - \mathsf{Per}(B_r \cap \Sigma)) \tag{10}$$

$$= \quad \left(2\pi r - \frac{2}{R}\pi r^2\right) + \left(\frac{2}{R}\pi\rho^2 - 2\pi\rho^*\right) \tag{11}$$

$$= \quad 2\pi r(1 - \frac{r}{R}) + 2\pi\rho(\frac{\rho}{R} - \frac{\rho^*}{\rho}) \tag{12}$$
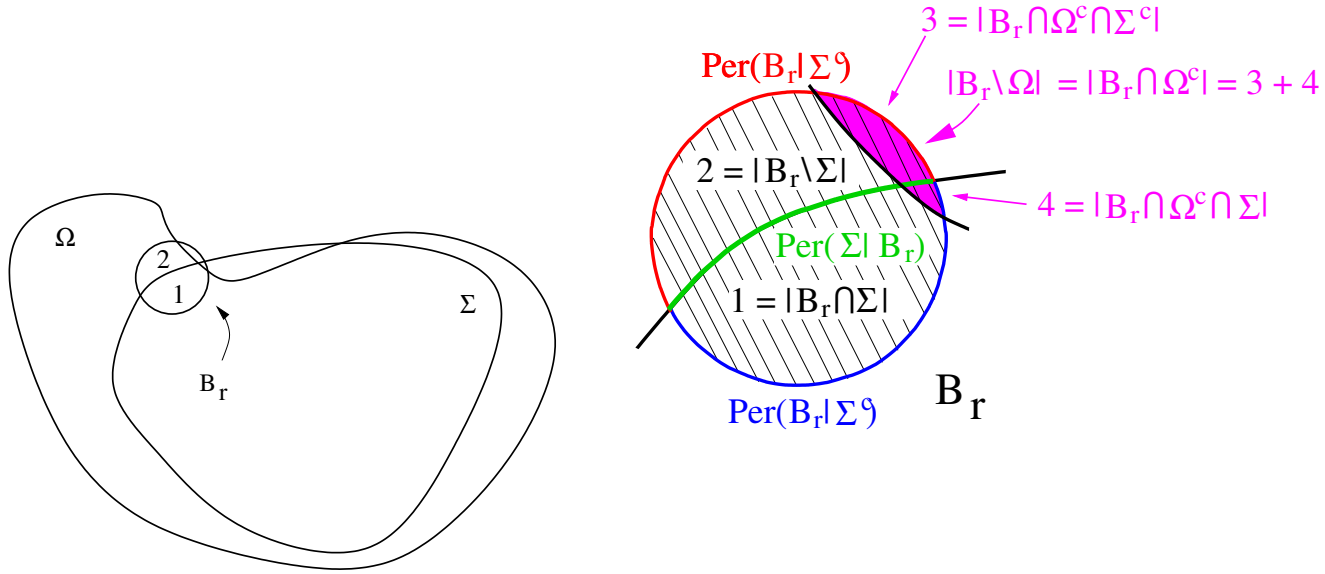
$$E(\Sigma \cup B_r) - E(\Sigma) \leq 2\pi r(1 - \frac{r}{R}) + 2\pi\rho(\frac{\rho}{R} - \frac{\rho^*}{\rho}) + 2\lambda|B_r \setminus \Omega| \qquad (13)$$

Theorem 2. *Given a ball $B_{\hat{r}}$ with $\frac{2}{\lambda} < \hat{r} < \frac{5}{\lambda}$ and an ϵ such that $(1 - \frac{1}{\sqrt{2}}) > \epsilon > 0$, we can choose a $\theta > 0$ such that if $|B_r \setminus \Omega| < \theta|B_{\hat{r}}|$, then $B_{r^*} \subset \Sigma$ for $r^* = (1-\epsilon)\frac{2}{\lambda}$.*
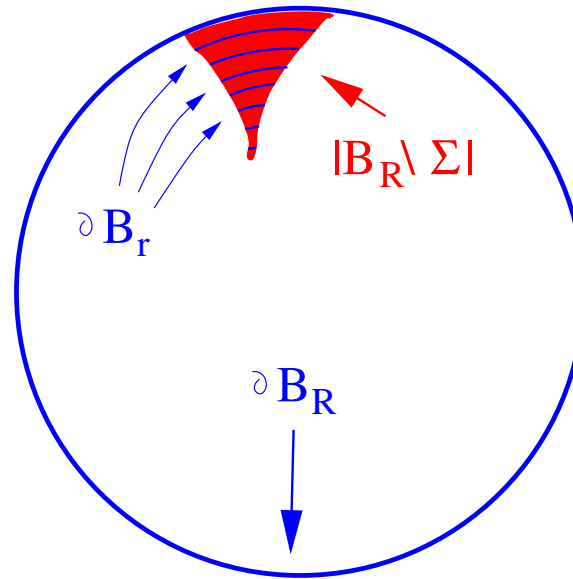
$$E(\Sigma \cup B_r) - E(\Sigma) \quad \leq \quad -\mathsf{Per}(\Sigma; B_r(x_0)) + \mathsf{Per}(B_r; \Sigma^c) + \lambda|B_r \setminus \Sigma| \quad (14)$$

$$\leq \quad -C\sqrt{v(r)} + \dot{v}(r) + \lambda v(r) \quad (v(r) \equiv |B_r \setminus \Sigma|) \quad (15)$$

$$0 \leq -C\sqrt{v(r)} + \dot{v}(r) + \lambda v(r) \Rightarrow \sqrt{v(r)} \leq \frac{CR}{2}\left(e^{-\frac{R-r}{R}} - 1\right) + \sqrt{|B_R \setminus \Sigma|}$$

$\Rightarrow$ small enough $|B_R \setminus \Sigma| \Rightarrow v(R - \epsilon) = 0 \Rightarrow B_{R-\epsilon} \subset \Sigma$.

$$|B_R \setminus \Omega| \text{ small} \stackrel{?}{\Rightarrow} |B_R \setminus \Sigma| \text{ small}$$

Idea of proof:

The rest of the proof is a fairly intricate argument showing that when:

$$|B_R \setminus \Omega| < \delta$$

then of the three cases:

1 $|B_R \setminus \Sigma| \leq N\delta$

2 $N\delta \leq |B_R \setminus \Sigma| < \frac{1}{4}\pi R^2$

3 $\frac{1}{4}\pi R^2 \leq |B_R \setminus \Sigma|$

only case 1 occurs. This is obtained by making use of:

$$E(\Sigma \cup B_r) - E(\Sigma) \leq 2\pi r(1 - \frac{r}{R}) + 2\pi\rho(\frac{\rho}{R} - \frac{\rho^*}{\rho}) + 2\lambda|B_r \setminus \Omega| \qquad (16)$$

Theorem 3. *Using a recent result of Allard's, we can conclude that for convex Ω, $\Sigma =$ the union of all $\frac{1}{\lambda}$ balls which are contained in Ω PROVIDED there is at least one $\frac{2}{\lambda}$ ball contained in Ω.*

Outline of Proof:

If Ω is convex and Σ (which must be contained in Ω) is not empty, then Σ is the union of the $\frac{1}{\lambda}$ balls in Ω. Our result says that if Ω contains a $\frac{2}{\lambda}$ ball, then it is contained in a solution Σ. Therefore, using Allard's result, Σ must equal the union of $\frac{1}{\lambda}$ balls in Ω.

Comments and Conclusions

- To Do: Establish connections to morphology – opening and closing, etc.

- To Do: Exact solutions with noise – further results.

- To Do: Understand the regularization and reconstruction aspects image analysis for experimental data in which physics is partly understood and partly being explored.